# EFFICIENT UPDATING OF MODEL PARAMETERS USING MODAL COORDINATES


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## ABSTRACT

Model correlation methods based on iterative modification of physical model parameters have been used with frequent success. The cost and time to perform these methods is driven mostly by the need to recompute normal modes and parameter sensitivities from the full finite element model after each adjustment. This paper presents a refinement of the method, in which multiple iterations can be performed before the full model must be run.


## NOMENCLATURE

| | |
|---|---|
| $[M]$ | physical mass matrix |
| $[K_0]$ | physical stiffness matrix, unmodified model |
| $\alpha_p$ | $p$'th structural parameter, dimensionless |
| $[K_p]$ | delta stiffness for the $p$'th structural parameter, in physical coordinates |
| $[\kappa_p]$ | delta stiffness for the $p$'th structural parameter, in modal coordinates |
| $[\kappa_p]_{jk}$ | entry in $j$'th row, $k$'th column of $[\kappa_p]$ |
| $[\Phi_0]$ | mass normalized mode shapes of unmodified model in physical coordinates |
| $[\lambda_0]$ | eigenvalues of unmodified model |
| $(\lambda_0)_j$ | $j$'th eigenvalue of unmodified model |
| $[\Phi]$ | mass normalized mode shapes of modified model, function of $\alpha_p$ |
| $[\lambda]$ | eigenvalues of modified model, function of $\alpha_p$ |
| $[\Psi]$ | mass normalized mode shapes of modified model in modal coordinates, function of $\alpha_p$ |
| $[\lambda_p]$ | Sensitivities of $[\lambda]$ to $\alpha_p$ |
| $[\Psi_p]$ | Sensitivics of $[\Phi]$ to $\alpha_p$ |

## 1. INTRODUCTION

A significant part of any modal test is the need to update finite element models to improve agreement with test measurements. A widely-used approach for model updating involves adjusting a number of physical model parameters in order to get a better test/analysis match. The change to model parameters is usually guided by first order sensitivity of mode frequencies and/or mode shapes to the parameters [1,2].

Unfortunately, the sensitivity calculation is valid only for small parameter changes. The frequencies and mode shapes estimated by linear sensitivity approximations quickly diverge from the exact frequencies and mode shapes obtained from solution of the eigenvalue problem. This means in practice that structural parameters must be modified iteratively. After a small change, the full finite element model must be re-analyzed to determine the true effect of the parameter adjustment on frequencies and/or mode shapes. If further improvement is need, the linear sensitivity matrices must be calculated for the modified model, and another iteration may be performed on the structural parameter adjustment. The repeated eigensolutions of the full model account for most of the cost and time involved in performing this type of model updating.

This paper presents an approach which allows larger variations of model parameters to be assessed before re-running the full finite element model. The idea is to perform new eigensolutions in modal coordinates of the unperturbed model. A number of small parameter changes can be accumulated before incorporating the change into the full model.

The new approach was implemented for the model correlation effort associated with the Shuttle Imaging Radar-C (SIR-C) modal test. The method worked very well in updating the finite element model to match test-measured modes.

## 2. PROBLEM FORMULATION

For simplicity, it will be assumed that only stiffness matrix modification is being performed. A number of dimensionless structural parameters $\alpha_p$ are allowed to wry, such that the unmodified structural model corresponds to $\alpha_p = 0$. The dependence of the physical stiffness matrix $[K]$ on these parameters will be assumed as

$$[K] = [K_0] + \sum_p \alpha_p [K_p]. \tag{1}$$

That is, the stiffness matrix is assumed to be linear in the structural parameters. The development that follows could be performed for any other known functional form as well. However, the linear relationship is often either the true functional form, or is a reasonable accurate approximation over a broad range of parameter values.

Any model updating approach based on physical structural parameters must begin with an approximation of the frequencies and/or mode shapes which result from parameter adjustment. That is, the eigenvalue problem

$$[K][\Phi] = [M][\Phi][\lceil \lambda \rfloor] \tag{2}$$

is to be be solved for the eigenvalues $\lceil \lambda \rfloor$ and mode shapes $[\Phi]$, with [K] given by equation (1). Only the lowest frequency modes are usually of interest. Normally, this problem is solved exactly for a particular set of parameters by using the full finite element model. In order to select a desirable parameter set, linear sensitivity analysis is used to estimate $\lceil \lambda \rfloor$ and $[\Phi]$ as functions of the $\alpha_p$.

The main ides of this paper is to solve equation (2) exactly without using the full finite element model. The solution will be exact, provided the stiffness matrix formulation of equation (1) is valid.

## 3. MODAL COORDINATES

The unmodified model has eigenvalues $\lceil \lambda_0 \rfloor$ and mass-normalized mode shapes $[\Phi_0]$, which solve the eigenvalue problem (2) with all parameters set to zero:

$$[K_0][\Phi_0] = [M][\Phi_0][\lceil \lambda \rfloor]. \tag{3}$$

$[\Phi_0]$ and $\lceil \lambda_0 \rfloor$ satisfy the orthogonality relationships

$$[\Phi_0]^T[M][\Phi_0] = \lceil I \rfloor, \tag{4}$$

$$[\Phi_0]^T[K_0] [\Phi_0] = \lceil \lambda_0 \rfloor. \tag{5}$$

Assume that the mode shapes for the modified model are linear combinations of the mode shapes of the unmodified model. That is,

$$[\Phi] = [\Phi_0][\Psi] \tag{6}$$

for some square matrix $[\Psi]$. Such a relationship is always true if $[\Phi_0]$ is the full (square) mode shape matrix, since in that case the columns of $[\Phi_0]$ span the full space of possible structural motions. This is only approximately true if not all columns of $[\Phi_0]$ are retained.

Substitution of equation (6) into the eigenvalue problem (2), and pre-multiplying by $[\Phi_0]^T$, we obtain the new eigenvalue problem

$$\left(\lceil \lambda_0 \rfloor + \sum_p \alpha_p [\kappa_p]\right)[\Psi] = [\Psi][\lceil \lambda \rfloor], \tag{7}$$

where

$$[\kappa_p] = [\Phi_0]^T[K_p][\Phi_0]. \tag{8}$$

Equation (7) is equivalent to the original eigenvalue problem of equation (2), except in modal coordinates the mass matrix is an identity, and the stiffness matrix is expressed in terms of the unmodified eigenvalues and the modal stiffness sensitivities defined in equation (8). Most importantly, the size of the eigenvalue problem in equation (7) is equal to the number of retained modes from the full model needed to span the space of the new mode shapes, whereas the original equation (2) is as large as the full model.

For any choice of structural parameters $\alpha_p$, the eigenvalue problem of equation (7) can be solved inexpensively for the full (square) mode shape matrix $[\Psi]$ and eigenvalues $\lceil \lambda \rfloor$. The only required information is: eigenvalues $\lceil \lambda_0 \rfloor$ of the unmodified model, and the stiffness sensitivities $[\kappa_p]$. Both of these can be generated easily from the full unmodified model. The mode shapes in physical coordinates are then given by equation (6). This process yields the exact solution for the modified model, provided enough modes of the unmodified model are retained in equation (6), and assuming equation (1) holds exactly.

## 4. SENSITIVITY COMPUTATIONS

Most algorithms for automatic or computer-aided parameter update are guided by first order (linear) sensitivity of frequencies or mode shapes to the structural parameters. These sensitivities have been derived elsewhere, but a simple derivation follows based on the eigenvalue problem in modal coordinates.

Let

$$\lceil \lambda_p \rfloor = \frac{\partial}{\partial \alpha_p} \lceil \lambda \rfloor \tag{9}$$

$$[\Psi_p] = \frac{\partial}{\partial \alpha_p} [\Psi]. \tag{10}$$

Then the mode shape sensitivity in physical coordinates is

$$\frac{\partial}{\partial \alpha_p} [\Phi] = [\Phi_0][\Psi_p]. \tag{11}$$

Now write the Taylor expansions

$$\lceil\lambda\rfloor = \lceil\lambda_0\rfloor + \sum_p \alpha_p \lceil\lambda_p\rfloor + O(\alpha_p^2), \tag{12}$$

$$[\Psi] = \lceil I\rfloor + \sum_p \alpha_p [\Psi_p] + O(\alpha_p^2) \tag{13}$$

The unknown linear sensitivities $\lceil\lambda_p\rfloor$ and $[\Psi_p]$ will be discovered based on the orthogonality relationships satisfied by $[\Psi]$:

$$[\Psi]^T[\Psi] = \lceil I\rfloor, \tag{14}$$

$$[\Psi]^T\Big(\lceil\lambda_0\rfloor + \sum_p \alpha_p[\kappa_p]\Big)[\Psi] = \lceil\lambda\rfloor. \tag{15}$$

Substituting equations (12) and (13) into equations (14) and (15), and collecting terms linear in $\alpha_p$, we find

$$\sum_p \alpha_p\Big([\Psi_p]^T + [\Psi_p]\Big) = \lceil 0\rfloor, \tag{16}$$

$$\sum_p \alpha_p\Big([\Psi_p]^T\lceil\lambda_0\rfloor + \lceil\lambda_0\rfloor[\Psi_p] + [\kappa_p] - \lceil\lambda_p\rfloor\Big) = \lceil 0\rfloor. \tag{17}$$

Since this must be true for arbitrary $\alpha_p$, equation (16) implies

$$[\Psi_p]^T = -[\Psi_p], \tag{18}$$

and this together with equation (17) means that

$$\lceil\lambda_0\rfloor[\Psi_p] - [\Psi_p]\lceil\lambda_0\rfloor + \lceil\lambda_p\rfloor = [\kappa_p]. \tag{19}$$

The $j$'th diagonal entry in the above matrix equation is

$$(\lambda_p)_j = [\kappa_p]_{jj}, \tag{20}$$

which means that the eigenvalue sensitivities are found on the diagonal of the stiffness sensitivity matrix in modal coordinates.

An off-diagonal entry in equation (19) in the $j$'th row and $k$'th column is

$$\Big((\lambda_0)_k - (\lambda_0)_j\Big)[\Psi_p]_{jk} = [\kappa_p]_{jk}, \quad \text{if } j \neq k. \tag{21}$$

In the normal case when the eigenvalues of the unmodified model are distinct, this equation can then be solved for the off-diagonal terms of the mode shape sensitivity matrix $[\Psi_p]$. The diagonal terms of $[\Psi_p]$ must be zero in order to satisfy equation (18). Once the $[\Psi_p]$ matrices are determined, the full mode shape sensitivities are obtained using equation (11).

Equation (21) breaks down when eigenvalues of the unmodified model approach each other. This suggests that the mode shape sensitivity becomes large for closely spaced modes. In practice, the eigenvalues will never be exactly equal, but the mode shapes predicted by linear sensitivity analysis are only valid for very tiny parameter changes in the case of closely spaced modes.

## 5. SENSITIVITY CALCULATION FOR MODIFIED MODEL

The above results are not new. On the surface, they merely suggest one way to derive previously published frequency and mode shape sensitivity results. The new idea of this paper is that after a small adjustment has been made to the structural parameters, the new modal properties of the modified model can be computed exactly, and then the sensitivities of the new frequencies and mode shapes can be computed, all without needing to revert to the finite element model.

Equation (7) shows the eigenvalue problem to be solved in order to obtain the eigenvalues $\lceil\lambda\rfloor$ and mass normalized mode shapes $[\Phi]$ of the modified model. Equations (20) and (21) show how to obtain the sensitivity of the eigenvalues and mode shapes to the structural parameters.

It is worth repeating that only the following information is required to calculate the exact eigenvalues and mode shapes, and both eigenvalue and mode shape sensitivities: eigenvalues $\lceil\lambda_0\rfloor$ of the unmodified model; mass normalized mode shapes $[\Phi_0]$ of the unmodified model (a sufficient subset of all modes); and the stiffness sensitivies $[\kappa_p]$ in modal coordinates of the unmodified model.

If a modification is made to the model, then we can perform all of the above calculations using the modified model as a new starting point. Of the three items listed above, the eigenvalues and mode shapes come directly from the solution of equation (7). That is, to make the modified model the new baseline, we simply replace

$$\lceil\lambda\rfloor \to \lceil\lambda_0\rfloor, \tag{22}$$

$$[\Phi_0][\Psi] \to [\Phi_0]. \tag{23}$$

And according to equation (8), the stiffness sensitivies in the new modal coordinates should be

$$[\Psi]^T[\kappa_p][\Psi] \to [\kappa_p]. \tag{24}$$

Note that this latter equation would need to be adjusted if a nonlinear stiffness relationship was used in place of equation (1).

With the above substitutions, the sensitivity calculation described in section 4 can be repeated to obtain sensitivities valid in the vicinity of the new model. Thus, a number of small steps can be taken without reverting to the full model. At each step, the new frequencies and mode shapes are computed without error, and the sensitivities are recomputed appropriately.

In practice, it may not be possible to completely avoid iterations using the finite element model. Errors will accumulate if equation (1) is not exact, or due to the limited number of modes of the original model being used in the calculation. However, this technique allows a large number of small steps to be accumulated before it is necessary to run the full model.

## 6. SUGGESTED MODEL UPDATING APPROACH

Using the above results, it is possible to perform parameter-based model updating in a systematic way. The following steps outline one possible approach.

1. Select structural parameters $\alpha_p$.

2. Using unmodified finite element model, compute eigenvalues $\lceil \lambda_0 \rfloor$ and mode shapes $[\Phi_0]$.

3. Using unmodified finite element model, compute stiffness sensitivities $[\kappa_p] = [\Phi_0]^T [K_p][\Phi_0]$.

4. Generate eigenvalue sensitivities $\lceil \lambda_p \rfloor$ and/or mode shape sensitivities $[\Phi_p] = [\Phi_0][\Psi_p]$.

5. Use sensitivities to select promising modifications to structural parameters. (A number of approaches have been suggested for choosing parameter changes. For example, minimize errors between test and analysis while keeping parameter changes small.) It is possible to interactively try different parameter changes, and use the exact eigenvalue solution to preview the result of each change.

6. After making a small change in parameters, solve the eigenvalue problem

$$\left(\lceil \lambda_0 \rfloor + \sum_p \alpha_p[\kappa_p]\right)[\Psi] = [\Psi]\lceil \lambda \rfloor$$

   for $\lceil \lambda \rfloor$ and $[\Psi]$. Make the modified model the new starting point try replacing

$$\lceil \lambda \rfloor \rightarrow \lceil \lambda_0 \rfloor$$
$$[\Phi_0][\Psi] \rightarrow [\Phi_0]$$
$$[\Psi]^T[\kappa_p][\Psi] \rightarrow [\kappa_p].$$

7. Starting from this new baseline, begin another iteration at step 4.

## 8. CONCLUSIONS

A technique was presented for using modal coordinates to facilitate multiple iterations of structural parameter modification without frequently exercising the full finite element model. After each iteration, an exact solution of the eigenvalue problem is performed, and linear sensitivity is recalculated.

This refinement of previously reported parameter modification methods allows iterations to be performed essentially real time. As a result, the analyst is free to experiment with many options for parameter adjustment, and can immediately preview the consequences of any change. In addition, it is more feasible to perform a larger number of small parameter changes, with the sensitivities recalculated after each change. This is especially useful when it is necessary to correlate a model with closely spaced modes.

## 9. ACKNOWLEDGEMENT

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## 10. REFERENCES

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